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1997 J. Phys. A: Math. Gen. 30 3249

(http://iopscience.iop.org/0305-4470/30/9/030)

J. Phys. A: Math. Gen. **30** (1997) 3249–3259. Printed in the UK PII: S0305-4470(97)76080-3

The duality relation between Glauber dynamics and the diffusion–annihilation model as a similarity transformation

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Received 2 July 1996

Abstract. In this paper we address the relationship between zero temperature Glauber dynamics and the diffusion–annihilation problem in the free fermion case. We show that the well known duality transformation between the two problems can be formulated as a similarity transformation if one uses appropriate (toroidal) boundary conditions. This allows us to establish and clarify the precise nature of the relationship between the two models. In this way we obtain a one-to-one correspondence between observables and initial states in the two problems. A random initial state in Glauber dynamics is related to a short-range correlated state in the annihilation problem. In particular, the long-time behaviour of the density in this state is seen to depend on the initial conditions. Hence, we show that the presence of correlations in the initial state determine the dependence of the long-time behaviour of the density on the initial conditions, even if such correlations are short ranged. We also apply a field-theoretical method to the calculation of multi-time correlation functions in this initial state.

1. Introduction

It has been known for a long time that there is a relation between Glauber dynamics [1] and the symmetric diffusion problem in the presence of annihilation and deposition of pairs of particles for a certain choice of the diffusion constant [2], which corresponds to the case in which this problem can be solved using free fermions [3, 4]. Using this relation, Family and Amar [5] have computed the time evolution of the particle density in the transformed state of the annihilation problem that corresponds to random initial conditions in Glauber dynamics at $T = 0$. They have shown that their result only agrees with the previously known results by Spouge [6], if one starts with zero initial magnetization in the Glauber problem. In all other cases the long-time behaviour of the density shows a dependence on the initial conditions, a result which differs from the well known universal behaviour, valid for random initial states in the annihilation problem [7]. This raises the question of the correspondence between initial states in the Glauber and annihilation problems and, more generally, the relation between observables in the two systems. In this paper, we show that the duality transformation between the two systems is really a similarity transformation, if one uses sector-dependent, toroidal boundary conditions in the Glauber model (see below). We show that a random initial state in Glauber dynamics is mapped through this similarity transformation to a state with nearest-neighbour correlations. Such a state is translationally invariant, which allows us to recover the result by Family and Amar

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and also to compute higher-order correlators using a field-theoretic technique. The relation between these quantities in the transformed state and their counterparts in the random initial state with density 1/2 is explored and some calculations are explicitly done. Finally, we emphasize the role of the correlations in the initial state for the failure of the universality hypothesis.

The structure of this paper is as follows: in section 2, we present Glauber dynamics in terms of a quantum spin chain and the definition of a Temperley–Lieb algebra in terms of the spin-$\frac{1}{2}$ operators. The associated Hecke algebra permits the construction of the similarity transformation to the reaction–diffusion model and also determines the boundary conditions of the system. The correspondence between observables in Glauber and annihilation dynamics is also addressed. In section 3, we discuss the relation between the initial states in the two problems. In particular, we study the mapping between the random initial state in Glauber dynamics and a short-range correlated state in annihilation dynamics. Using these results and those of section 2 we establish a correspondence between correlation functions in the two problems. In section 4, we show that the short-range correlated state is a translationally invariant state and we show how to calculate multiple-time correlation functions in this state using a field-theoretic technique. The density is explicitly computed and shown to agree with Family and Amar's result. We study the relation between two-point correlation functions in these states and the random initial state with density 1/2 and recover the zero time correlations as a special case. Finally, in section 5, we present our conclusions.

For simplicity we discuss only the case of zero temperature Glauber dynamics although our results are easily generalized to other problems with little modification, for example, Glauber dynamics at finite temperature or the model of generalized dynamics considered by Peschel and Emery [8], which maps by a duality transformation to a model of diffusing particles with pair annihilation and creation away from the free fermion line.

2. The transformation law for the operators

It is well known [3, 9] that certain reaction–diffusion systems provide physical realizations of Hecke algebras. The time evolution of these processes is described by a master equation which can be conveniently written using an operator formalism, in which one assigns to each configuration of the system a state vector in an Hilbert space [10]. The probability distribution is then represented by a state vector obeying a 'Schrödinger' equation $\partial_t|\Psi\rangle = -H|\Psi\rangle$ with $|\Psi\rangle = \sum_{\underline{n}} P(\underline{n}, t)\underline{n}\rangle$ where $P(\underline{n}, t)$ is the probability of finding configuration $\underline{n}$ at time t and the set of different $|\underline{n}\rangle$ is supposed to be orthonormal and complete. The operator H is a linear and in general non-Hermitian operator encoding the rules of the stochastic process. For some systems of interest this operator can be written [3, 9] as a sum of generators of Hecke algebras. Given that there exists a relation between reaction–diffusion systems and Glauber dynamics [2], it is natural to ask if the evolution operator for this system can also be written in terms of Hecke algebra generators, and if so, what conclusions can one draw from it. In order to show this we define the following operators on a lattice of L sites

$$
\begin{aligned}
H^{\pm} &\equiv \sum_{j=1}^{L}(1 - e_{2j-1})(1 - (e_{2j} + e_{2j-2} - 1)) \\
e_{2j-1} &= \tfrac{1}{2}(1 + \hat{\sigma}_j^x) \qquad 1 \leqslant j \leqslant L \\
e_{2j} &= \tfrac{1}{2}(1 + \hat{\sigma}_j^z\hat{\sigma}_{j+1}^z) \qquad 1 \leqslant j \leqslant L - 1
\end{aligned}
\tag{1}
$$

and $e_0 = e_{2L}$ which will be given below. The operators $\hat{\sigma}_j^z$, $\hat{\sigma}_j^x$ are Pauli spin matrices at site j.

The set $e_j (1 \leqslant j \leqslant 2L-1)$ forms a Temperley–Lieb algebra [11], characterized by the relations

$$\begin{aligned} e_j^2 &= e_j \\ e_j e_{j\pm1} e_j &= \tfrac{1}{2} e_j \\ e_j e_i &= e_i e_j \qquad \text{if } |j-i| \geqslant 2. \end{aligned} \tag{2}$$

In order to construct e_{2L} explicitly we define the set of operators $g_j (1 \leqslant j \leqslant 2L-1)$ and the duality operator D by [12]

$$\begin{aligned} g_j &= (1+i)e_j - 1 \\ D &= \left(\prod_{j=1}^{2L-1} g_j \right) X \end{aligned} \tag{3}$$

where $\prod_{i=1}^{2L-1} g_j$ is the ordered product of the g_js and X will be either 1 or $\hat{\sigma}_L^z$ in this paper. The important points about these two operators are that both commute with g_j for $1 \leqslant j \leqslant 2L-2$ and that $(g_{2L-1}X)^2 = (Xg_{2L-1})^2$. When $X = 1$, we will call the corresponding duality operator D_+ and when $X = \hat{\sigma}_L^z$, we will call it D_-. As we will see below the choice of the operator D is directly related to the different types of toroidal boundary conditions that were referred in the last section. The operators g_j, together with the commutation relations with X, form an affine Hecke algebra associated with the Temperley–Lieb algebra given above (see [3, 9, 12] and references therein) and one finds $De_j = e_{j+1}D$ for $1 \leqslant j \leqslant 2L-2$. Since $g_j^\dagger = (1-i)e_j - 1$, one finds from (3) that $g_j^\dagger g_j = 1$. Also $X^2 = 1$, and we conclude that D is unitary and hence invertible. We define e_{2L} as

$$e_{2L} = De_{2L-1}D^{-1}. \tag{4}$$

The set of operators $e_j (1 \leqslant j \leqslant 2L-1)$ and e_{2L} satisfies the relations of a periodic Temperley–Lieb algebra [12] with $2L$ generators which is defined by (2) together with similar relations for $e_{2L} (e_{2L+1} = e_1$, etc). For the particular choices of X given above, it can be shown that [13]

$$e_{2L} = \begin{cases} \frac{1}{2}(1 + \hat{C}\hat{\sigma}_L^z\hat{\sigma}_1^z) & \text{if } X = 1 \\ \frac{1}{2}(1 - \hat{C}\hat{\sigma}_L^z\hat{\sigma}_1^z) & \text{if } X = \hat{\sigma}_L^z \end{cases} \tag{5}$$

where $\hat{C} = \prod_{j=1}^{L} \hat{\sigma}_j^x$. If we substitute the definitions of the e_js in (1) $H^\pm$ can be seen to be the generator of the time evolution for Glauber dynamics. The $+$ $(-)$ sign stands when $X = 1$ $(X = \hat{\sigma}_L^z)$. This explains the use of the notation $H^\pm$. The choice $X = 1$ $(X = \hat{\sigma}_L^z)$, corresponds to the operator H^+ (H^-) which, when applied to eigenstates of $\hat{C}$ with eigenvalue 1 (-1), generates the time evolution for Glauber dynamics at $T = 0$ with periodic boundary conditions [14]. When applied to eigenstates of $\hat{C}$ with eigenvalue -1 (1) H^+ (H^-) generates the time evolution for Glauber dynamics with anti-periodic boundary conditions. Both dynamics are stochastic. We see that one can indeed write the evolution operator for Glauber dynamics in terms of Hecke algebra generators. We now define the following similarity transformation

$$V_\pm = RD_\pm \tag{6}$$

where $R = \exp(\mathrm{i}(\pi/4)\sum_{j=1}^{L}\hat{\sigma}_j^y)$ is a global rotation of $\pi/2$ around the y-axis. From the relations between the Temperley–Lieb generators $e_{j+1} = De_jD^{-1}$ it can be easily shown that

$$
\begin{aligned}
&V_\pm \hat{\sigma}_j^x V_\pm^{-1} = \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x \qquad 1 \leqslant j \leqslant L-1 \\
&V_\pm \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z V_\pm^{-1} = \hat{\sigma}_{j+1}^z \qquad 1 \leqslant j \leqslant L-1 \\
&V_\pm \hat{\sigma}_L^x V_\pm^{-1} = \pm \hat{Q}_L \hat{\sigma}_L^x \hat{\sigma}_1^x \\
&V_\pm \hat{C} \hat{\sigma}_L^z \hat{\sigma}_1^z V_\pm^{-1} = \pm \hat{\sigma}_1^z
\end{aligned}
\tag{7}
$$

where $\hat{Q}_L = R\hat{C}R^{-1} = \prod_{j=1}^{L}\hat{\sigma}_j^z$. From (7) it also follows that $V_\pm \hat{C} V_\pm^{-1} = \pm\hat{Q}_L$. Applying V_+ to H^+ and V_- to H^- gives

$$
\begin{aligned}
\tilde{H}^\pm = V_\pm H^\pm V_\pm^{-1} = &\sum_{j=2}^{L-1} \frac{1}{2}(1 - \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x)\left(1 - \frac{1}{2}(\hat{\sigma}_j^z + \hat{\sigma}_{j+1}^z)\right) \\
&+ \frac{1}{2}(1 - \hat{\sigma}_1^x \hat{\sigma}_2^x)\left(1 - \frac{1}{2}(\hat{\sigma}_1^z + \hat{\sigma}_2^z)\right) + \frac{1}{2}(1 \mp \hat{Q}_L \hat{\sigma}_L^x \hat{\sigma}_1^x)\left(1 - \frac{1}{2}(\hat{\sigma}_1^z + \hat{\sigma}_L^z)\right).
\end{aligned}
\tag{8}
$$

The operators $\tilde{H}^\pm$, restricted to the subspaces $\hat{Q}_L = 1$ for $\tilde{H}^+$ and $\hat{Q}_L = -1$ for $\tilde{H}^-$, are equivalent to the Hamiltonian of the diffusion–annihilation problem with rates of diffusion $1/2$ and rate of annihilation 1 and periodic boundary conditions, which can be solved in terms of free fermions [3]. The other cases correspond to non-stochastic processes. Hence, we obtain a rigorous formulation of the well known duality transformation between the two models [2]. These results are summarized in table 1. Notice that, although we have not used it explicitly, the similarity transformation preserves the relations (2) so one can also represent $\tilde{H}^\pm$ in terms of Hecke algebra generators. However, one has used here an Hermitian quotient of the algebra which is different from the one used in [3, 9] and which allows the Hamiltonian for the diffusion–annihilation to be written as a linear combination of Hecke algebra generators.

Finally, let us consider the action of $V_\pm$ in a single $\hat{\sigma}_j^z$ operator. Expressing the operators g_l in terms of Pauli spin matrices in (3) one can, using the commutation relations for these operators, show that

$$
V_\pm \hat{\sigma}_j^z V_\pm^{-1} = -\hat{\sigma}_1^y \hat{\sigma}_2^z \ldots \hat{\sigma}_j^z. \tag{9}
$$

Using (9), one obtains the following transformation law for a pair of $\hat{\sigma}_k^z\hat{\sigma}_l^z k < l$

$$
V_\pm \hat{\sigma}_k^z \hat{\sigma}_l^z V_\pm^{-1} = (V_\pm \hat{\sigma}_k^z V_\pm^{-1})(V_\pm \hat{\sigma}_l^z V_\pm^{-1}) = (\hat{\sigma}_1^y \hat{\sigma}_2^z \ldots \hat{\sigma}_k^z)(\hat{\sigma}_1^y \hat{\sigma}_2^z \ldots \hat{\sigma}_k^z \ldots \hat{\sigma}_l^z) = \hat{\sigma}_{k+1}^z \ldots \hat{\sigma}_l^z. \tag{10}
$$

Note, that taking $l = k+1$ or $k = L$, $l = 1$ we recover the equalities (7) concerning $\hat{\sigma}_{k+1}^z$. Equation (10) will be useful below when we derive equalities concerning correlation functions. We now proceed to study the effect of $V_\pm$ in the states of the theory.

3. The transformation law for the states

The Glauber–Ising Hamiltonian at $T = 0$ has two ground states, the ferromagnetic states with all spins up or down. Since H^+ is only equivalent to it in the subspace of the states with $\hat{C} = 1$, we have to find a linear combination of these two states that belong to this subspace (for simplicity we will specialize in V_+). This state is simply $|\Psi\rangle = \frac{1}{2}(|\uparrow\ldots\uparrow\rangle + |\downarrow\ldots\downarrow\rangle)$. It is normalized in the sense that $\langle s|\Psi\rangle = 1$, where $\langle s|$ is the sum of all configurations of spins with weight one. This expresses the fact that, for a stochastic process, the sum of the

Table 1. Summary of the relations between Glauber and annihilation dynamics. On the left-hand side we have the Hamiltonian operator which is equivalent to the Glauber–Ising Hamiltonian and the sector of the Hilbert space where that equivalence holds, indicated by the eigenvalue of $\hat{C}$. On the right-hand side we have the tranformed Hamiltonian and the sector in which it is equivalent to the diffusion–annihilation process (given by the eigenvalue of $\hat{Q}_L$).

Glauber dynamics		Annihilation dynamics	
Operators	Boundary conditions	Operators	Boundary conditions
H^+, $\hat{C} = 1$	Periodic	$\tilde{H}^+$, $\hat{Q}_L = 1$	Periodic
H^-, $\hat{C} = 1$	Antiperiodic	$\tilde{H}^-$, $\hat{Q}_L = -1$	Periodic
H^+, $\hat{C} = -1$	Antiperiodic	$\tilde{H}^+$, $\hat{Q}_L = -1$	Non-physical
H^-, $\hat{C} = -1$	Periodic	$\tilde{H}^-$, $\hat{Q}_L = 1$	Non-physical

probabilities of the different configurations accessible to the system (e.g. the configurations of spins in the Glauber problem) has to add to one. Also if H is the stochastic Hamiltonian representing the dynamics then it follows from conservation of probability that $\langle s|$ is the left ground state of H, i.e. $\langle s|H = 0$. One can easily check that $\hat{C}|\Psi\rangle = |\Psi\rangle$. This equality implies that

$$V_+\rangle\Psi| = V_+\hat{C}|\Psi\rangle = (V_+\hat{C}V_+^{-1})V_+|\Psi\rangle = \hat{Q}_L V_+|\Psi\rangle \tag{11}$$

where we have used the transformation law for $\hat{C}$ found above. Thus, the transformed state $V_+|\Psi\rangle$ belongs to the eigenspace with $\hat{Q}_L = 1$. In the annihilation language this corresponds to the sector with an even number of particles [4]. Also, since $H^+|\Psi\rangle = 0$, one obtains

$$V_+H^+|\Psi\rangle = (V_+H^+V_+^{-1})V_+|\Psi\rangle = \tilde{H}^+V_+|\Psi\rangle = 0 \tag{12}$$

and hence $V_+|\Psi\rangle$ is a ground state of the annihilation Hamiltonian (in the subspace $\hat{Q}_L = 1$, $\tilde{H}^+$ is equivalent to it). The only ground state belonging to the subspace with an even number of particles is the vacuum $|0\rangle$, i.e. the state with no particles. Hence, we conclude that $V_+|\Psi\rangle \propto |0\rangle$. The proportionality constant can be shown, using (3), to be equal to $(i/\sqrt{2})(-1)^{L-1}\,\mathrm{e}^{\mathrm{i}(\pi/4)(L-1)}$ and can be absorbed in the definition of V_+. Following the same reasoning that led to (12), and in light that $\langle s|\hat{C} = \langle s|$, one can similarly show that $\langle s|V_+^{-1}\tilde{H}^+ = 0$ and that $\langle s|V_+^{-1}$ belongs to the even sector. On the same grounds of uniqueness this shows that this state is equal (up to a normalization constant which we absorb in the definition of V_+^{-1}) to the left ground state of the annihilation Hamiltonian with an even number of particles, i.e. $\langle s|^{\mathrm{even}}$, which is the sum of all configurations with an even number of particles with weight one. We are using the same notation $\langle s|$ for the left ground states of the two Hamiltonians because both describe stochastic processes and this is the usual convention.

We have also found that H^- is equivalent to the Glauber–Ising Hamiltonian in the subspace $\hat{C} = -1$. Since $V_-\hat{C}V_-^{-1} = -\hat{Q}_L$ this subspace is mapped to the even sector of the annihilation problem. But in this sector $\tilde{H}^-$ (8) is not equivalent to a stochastic Hamiltonian. If one starts with the subspace $\hat{C} = 1$ then V_- effectively maps this sector to the odd sector where $\tilde{H}^-$ is equivalent to the annihilation Hamiltonian, but in this case H^- is not equivalent to the Glauber–Ising Hamiltonian with periodic boundary conditions, but to the Glauber–Ising Hamiltonian with anti-periodic boundary conditions, which is also a stochastic process. Using the same argument as above one can show that the ground state of this Hamiltonian is mapped to the ground state of the odd sector of the annihilation

Hamiltonian. This state is just a uniform superposition of states with one particle at each site of the lattice [4]. If we apply V_-^{-1} to this state we will therefore obtain the ground state of the Glauber–Ising Hamiltonian with anti-periodic boundary conditions (the observation about normalization factors also applies here). Such state is a uniform superposition of $2L-2$ states with two domain walls, one at the boundary and one at each site of the lattice, plus the two states with all spins up or down. These two states are, due to the anti-periodic boundary conditions, the image of the state with one particle at the boundary in the diffusion–annihilation model. The previous discussion shows that the transformations V_+ and V_- restricted to the $\hat{C}=1$ subspace, map a stochastic process (Glauber–Ising dynamics with periodic or anti-periodic boundary conditions) to a stochastic process (annihilation dynamics), something which we also pointed out in table 1. In the other cases the mapping is from a stochastic problem to a non-stochastic problem. Nevertheless, these mappings can be considered from a purely formal point of view and examples of similarity transformations between stochastic and non-stochastic processes have already been studied in the literature [14–16].

Here we shall concentrate in the stochastic–stochastic mapping given by V_+. We will consider the study of time-dependent correlation functions in uncorrelated random initial states evolving in time according to Glauber dynamics. The state

$$|\Phi\rangle = \prod_{j=1}^{L}\left[\frac{1+m}{2}+\frac{1-m}{2}\hat{\sigma}_j^x\right]|\Psi\rangle \tag{13}$$

corresponds to the superposition of two random initial states with initial magnetization m and $-m$. From the point of view of the calculation of correlation functions of an even number of $\hat{\sigma}_j^z$ operators the two states are equivalent and $|\Phi\rangle$ belongs to the $\hat{C}=1$ subspace. Under the application of V_+, $|\Phi\rangle$ transforms to

$$|\tilde{\Phi}\rangle = \prod_{j=1}^{L}\left[\frac{1+m}{2}+\frac{1-m}{2}\hat{\sigma}_j^x\hat{\sigma}_{j+1}^x\right]|0\rangle \tag{14}$$

where we have used (7) and the fact that $V_+|\Psi\rangle = |0\rangle$. This is an initial state with short-range correlations and its form will play a crucial role in the determination of the correlation functions in the late time regime. Under V_+ the multiple-time correlation functions of an even number of $\hat{\sigma}_j^z$ spins at times t_1, t_2, etc, transform as

$$\begin{aligned}
\langle s|&\hat{\sigma}_{j_1}^z(t_1)\hat{\sigma}_{j_2}^z(t_1)\hat{\sigma}_{j_3}^z(t_2)\hat{\sigma}_{j_4}^z(t_2)\ldots\hat{\sigma}_{j_{2N-1}}^z(t_N)\hat{\sigma}_{j_{2N}}^z(t_N)|\Phi\rangle\\
&= \langle s|\hat{\sigma}_{j_1}^z\hat{\sigma}_{j_2}^z\,\mathrm{e}^{-H^+(t_1-t_2)}\hat{\sigma}_{j_3}^z\hat{\sigma}_{j_4}^z\,\mathrm{e}^{-H^+(t_2-t_3)}\ldots\,\mathrm{e}^{-H^+(t_{N-1}-t_N)}\hat{\sigma}_{j_{2N-1}}^z\hat{\sigma}_{j_{2N}}^z\,\mathrm{e}^{-H^+t_N}|\Phi\rangle\\
&= \langle s|^{\mathrm{even}}\hat{\sigma}_{j_1+1}^z\ldots\hat{\sigma}_{j_2}^z\,\mathrm{e}^{-\tilde{H}^+(t_1-t_2)}\hat{\sigma}_{j_3+1}^z\ldots\hat{\sigma}_{j_4}^z\,\mathrm{e}^{-\tilde{H}^+(t_2-t_3)}\ldots\\
&\quad\ldots\,\mathrm{e}^{-\tilde{H}^+(t_{N-1}-t_N)}\hat{\sigma}_{j_{2N-1}+1}^z\ldots\hat{\sigma}_{j_{2N}}^z\,\mathrm{e}^{-\tilde{H}^+t_N}|\tilde{\Phi}\rangle\\
&= \langle s|\hat{\sigma}_{j_1+1}^z(t_1)\ldots\hat{\sigma}_{j_2}^z(t_1)\hat{\sigma}_{j_3+1}^z(t_2)\ldots\hat{\sigma}_{j_4}^z(t_2)\ldots\hat{\sigma}_{j_{2N-1}+1}^z(t_N)\ldots\hat{\sigma}_{j_{2N}}^z(t_N)|\tilde{\Phi}\rangle
\end{aligned} \tag{15}$$

where we have used equations (8) and (10). We suppose that $j_1 < j_2$, $j_3 < j_4$, etc. Also one concludes that for any odd number of $\hat{\sigma}_j^z$ operators, one has

$$\langle s|\hat{\sigma}_{j_1}^z(t_1)\ldots\hat{\sigma}_{j_{2N+1}}^z(t_{2N+1})|\Phi\rangle = 0 \tag{16}$$

since $\hat{\sigma}_j^z$ anticommutes with $\hat{C}$.

4. The calculation of correlation functions

Using (15) we have, in particular, that

$$\langle s|\tfrac{1}{2}(1-\hat{\sigma}^z_{j-1}(t)\hat{\sigma}^z_j(t))|\Phi\rangle = \langle s|\tfrac{1}{2}(1-\hat{\sigma}^z_j(t))|\tilde{\Phi}\rangle = \langle s|\hat{n}_j(t)|\tilde{\Phi}\rangle. \tag{17}$$

The operator $\frac{1}{2}(1-\hat{\sigma}^z_{j-1}\hat{\sigma}^z_j)$ checks for the existence of a domain wall at site j in the Glauber problem, i.e. it checks if at site j the spins cease to point up and start to point down or *vice versa*. The operator $\frac{1}{2}(1-\hat{\sigma}^z_j)$ checks for the existence of a particle at site j, i.e. if spin j is down in the annihilation problem [4]. This indeed corresponds to the well known duality transformation [2, 5]. Similar relations hold for higher-order correlation functions. Now we will use the fact that the diffusion annihilation Hamiltonian can be completely solved in terms of free fermions by means of the Jordan–Wigner transformation [17]. Using it, one is able to write the spin raising and lowering operators $\hat{s}^+_j$, $\hat{s}^-_j$ ($\hat{s}^{\pm}_j = \frac{1}{2}(\hat{\sigma}^x_j \pm \mathrm{i}\hat{\sigma}^y_j)$) in terms of creation and annihilation operators $\hat{a}^\dagger_j$, $\hat{a}_j$ for spinless fermions. It turns out that the calculation of a multiple-time correlation function of the $\hat{n}_j$ operators in state $|\tilde{\Phi}\rangle$ ($\hat{n}_j = \hat{a}^\dagger_j\hat{a}_j$ in the fermion language) can be reduced to the calculation of objects like $\langle s|\hat{b}_{p_1}\dots\hat{b}_{p_{2k}}|\tilde{\Phi}\rangle$ ($k \leqslant N$) [18], where $\hat{b}_p$ is the Fourier transform of the annihilation operator $\hat{a}_i$. The momentum labels p_j are half-odd integers between $-L/2+1$ and $L/2$ [3, 4]. Also, one is able to express the state (14) in terms of fermion operators. The result is

$$\begin{aligned}|\tilde{\Phi}\rangle &= \prod_{j=1}^{L}\left[\frac{1+m}{2} + \frac{1-m}{2}\hat{\sigma}^x_j\hat{\sigma}^x_{j+1}\right]|0\rangle = \exp\left(\beta\sum_{j=1}^{L}(\hat{\sigma}^x_j\hat{\sigma}^x_{j+1}-1)\right)|0\rangle \\ &= \exp\left(\beta\sum_{j=1}^{L}(\hat{a}^\dagger_j\hat{a}^\dagger_{j+1} + \hat{a}^\dagger_j\hat{a}_{j+1} + \hat{a}^\dagger_{j+1}\hat{a}_j + \hat{a}_{j+1}\hat{a}_j - 1)\right)|0\rangle\end{aligned} \tag{18}$$

where $m = \mathrm{e}^{-2\beta}$. The first equality follows from the fact that $(\hat{\sigma}^x_j\hat{\sigma}^x_{j+1})^2 = 1$ and the second just follows from the rules of the Jordan–Wigner transformation. In terms of the momentum space operators we can write $|\tilde{\Phi}\rangle$ as

$$\begin{aligned}|\tilde{\Phi}\rangle = \exp\Bigg(2\beta\sum_{p>0}\Bigg[&\cos\left(\frac{2\pi p}{L}\right)(\hat{b}^\dagger_p\hat{b}_p + \hat{b}^\dagger_{-p}\hat{b}^\dagger_{-p}) \\ &+\sin\left(\frac{2\pi p}{L}\right)(\hat{b}_p\hat{b}_{-p} + \hat{b}^\dagger_{-p}\hat{b}^\dagger_p) - 1\Bigg]\Bigg)|0\rangle.\end{aligned} \tag{19}$$

The simplest approach to use if one wants to calculate the density or any equal-time correlators is given in [4]. For the calculation of multiple-time correlators, we will follow a different route. We will look for operators $\hat{c}^\dagger_p$, $\hat{c}_p$ that diagonalize the quadratic form appearing in the exponent of (19). This can be accomplished by means of a Bogoliubov transformation [19]

$$\begin{aligned}\hat{c}_p &= \cos\left(\frac{\pi p}{L}\right)\hat{b}_p - \sin\left(\frac{\pi p}{L}\right)\hat{b}^\dagger_{-p} \\ \hat{c}^\dagger_p &= \cos\left(\frac{\pi p}{L}\right)\hat{b}^\dagger_p - \sin\left(\frac{\pi p}{L}\right)\hat{b}_{-p}\end{aligned} \tag{20}$$

and one gets

$$|\tilde{\Phi}\rangle = \exp\left(2\beta\sum_p(\hat{c}^\dagger_p\hat{c}_p - 1)\right)|0\rangle. \tag{21}$$

One then expands the exponential using the anti-commutation relations for the $\hat{c}_p^\dagger$, $\hat{c}_p$. Re-expressing these operators in terms of $\hat{b}_p^\dagger$, $\hat{b}_p$ and applying them to the vacuum, one finally obtains

$$|\tilde{\Phi}\rangle = \mathrm{e}^{-\beta L} \prod_{p>0} [\gamma_p + \delta_p \hat{b}_{-p}^\dagger \hat{b}_p^\dagger]|0\rangle \tag{22}$$

where $\gamma_p = \cosh(2\beta) - \sinh(2\beta)\cos(2\pi p/L)$ and $\delta_p = \sinh(2\beta)\sin(2\pi p/L)$. The form (22) expresses the translation invariance of the state $|\tilde{\Phi}\rangle$. One defines the following object

$$Z[\eta_p, \eta_{-p}] = \langle 0| \exp\left(\sum_{p>0}\left(\cot\left(\frac{\pi p}{L}\right)\hat{b}_p\hat{b}_{-p} + \eta_p\hat{b}_p + \eta_{-p}\hat{b}_{-p}\right)\right)|\tilde{\Phi}\rangle \tag{23}$$

where the quantities η_p, η_{-p} are Grassmann variables anti-commuting among themselves and with the $\hat{b}_p$s. Their presence is necessary to make the terms in the exponential commute with each other. It can be shown [18] that $Z[\eta_p, \eta_{-p}]$ is the generating function for the quantities $\langle s|\hat{b}_{p_1} \ldots \hat{b}_{p_{2k}}|\tilde{\Phi}\rangle$ that is

$$\langle s|\hat{b}_{p_1} \ldots \hat{b}_{p_{2k}}|\tilde{\Phi}\rangle = \partial_{\eta_{p_1}} \ldots \partial_{\eta_{p_{2k}}} Z[\eta_p, \eta_{-p}]|_{\eta_p=0}. \tag{24}$$

Substituting (22) in (23) we find, after a few algebraic manipulations, involving the anti-commutation relations between the $\hat{b}_p^\dagger$, $\hat{b}_p$ [18], the following expression for $Z[\eta_p, \eta_{-p}]$:

$$Z[\eta_p, \eta_{-p}] = \exp\left(\sum_{p>0} \sinh(2\beta)\,\mathrm{e}^{-2\beta} \sin\left(\frac{2\pi p}{L}\right)\eta_{-p}\eta_p\right). \tag{25}$$

Since $W = \ln Z$ is a quadratic function in the ηs it immediately follows that a Wick's decomposition holds for the quantities $\langle s|\hat{b}_{p_1} \ldots \hat{b}_{p_{2k}}|\tilde{\Phi}\rangle$ [18]. In particular, from (24) and (25) one has

$$\langle s|\hat{b}_{p'}\hat{b}_p|\tilde{\Phi}\rangle = \sinh(2\beta)\,\mathrm{e}^{-2\beta} \sin\left(\frac{2\pi p'}{L}\right)\delta_{p,-p'}. \tag{26}$$

Given that the expression for the space-dependent average density is [4]

$$\langle \hat{n}_j(t)\rangle = \frac{1}{L}\sum_{p,p'} \mathrm{e}^{(2\pi i/L)j(p-p')-(\epsilon_{-p}+\epsilon_{p'})t} \cot\left(\frac{\pi p}{L}\right)\langle s|\hat{b}_{p'}\hat{b}_{-p}|\tilde{\Phi}\rangle \tag{27}$$

where $\epsilon_p = 1 - \cos p$, one obtains, substituting (26) in (27) and taking the thermodynamic limit $L \to \infty$, the following expression for the density of particles $\rho(t)$ at time t

$$\rho(t) = \tfrac{1}{2}(1-m^2)\,\mathrm{e}^{-2t}(I_0(2t) + I_1(2t)) \tag{28}$$

where $I_0(2t)$, $I_1(2t)$ are the modified Bessel functions of order zero and one, and where we have used the identity $m = \mathrm{e}^{-2\beta}$. This is the well known expression obtained by Family and Amar [5] who have also considered a random initial state in Glauber dynamics. They have shown that while the initial distribution of spins is uncorrelated the distribution of domain walls, i.e. the distribution of particles in the annihilation problem, is correlated. This correlated structure is found in the transformed state $|\tilde{\Phi}\rangle$ (14). For $m = 0$, the expression (28) is identical to the one found by Spouge [6] for an uncorrelated random initial state with initial density 1/2 in the annihilation problem. Indeed, this identity is more general. If $m = 0$ then this means that we have to take the limit $\beta \to \infty$. If we take such a limit in equation (22), then one obtains the exact expression for a random initial state with density 1/2, projected over the even sector [4]. Therefore, our calculations provide a rigorous framework for the well known duality transformation. In the long-time limit $t \to \infty$ one finds from (28) the leading behaviour $\rho(t) \approx (1-m^2)/2\sqrt{\pi t}$. It depends on the initial conditions (i.e. magnetization) [5]. The amplitude of $\rho(t)$ differs from the universal result found for uncorrelated random initial states [7, 15]. Thus it is seen that the presence of correlations in the initial state breaks the universality of the amplitudes of correlation functions.

Our results, namely the Wick decomposition, also allow the explicit computation of higher-order correlation functions. Apart from a factor $2\sinh(2\beta)\,\mathrm{e}^{-2\beta}$ the result (26) is identical to the results obtained for the random initial state with density 1/2 [18] and, as stated above, it reduces to it when $m = 0$. Therefore, one has

$$\langle s|\hat{b}_{p_1}\ldots\hat{b}_{p_{2k}}|\tilde{\Phi}\rangle = (1-\mathrm{e}^{-4\beta})^k\langle s|\hat{b}_{p_1}\ldots\hat{b}_{p_{2k}}|1/2\rangle^{\mathrm{even}}. \tag{29}$$

As an example of the above result let us consider the two-point correlation function $\langle s|\hat{n}_j(t)\hat{n}_k(t')|\tilde{\Phi}\rangle$. We use (29) with $k = 2$ and $k = 1$ because this object can be written as a linear combination of the terms $\langle s|\hat{b}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}|\tilde{\Phi}\rangle$ and $\langle s|\hat{b}_{p_1}\hat{b}_{p_2}|\tilde{\Phi}\rangle$ [18], which contribute with different powers of $(1-\mathrm{e}^{-4\beta})$. Separating such powers, we obtain

$$\begin{aligned}\langle s|\hat{n}_j(t)\hat{n}_k(t')|\tilde{\Phi}\rangle &= (1-\mathrm{e}^{-4\beta})^2\langle s|\hat{n}_j(t)\hat{n}_k(t')|1/2\rangle^{\mathrm{even}} + \frac{1}{2L^2}[(1-\mathrm{e}^{-4\beta}) - (1-\mathrm{e}^{-4\beta})^2]\\ &\times\left\{\sum_{p_1,p_2}\mathrm{e}^{(2\pi\mathrm{i}/L)(p_1-p_2)(j-k)-(\epsilon_{-p_1}+\epsilon_{p_2})t+(\epsilon_{-p_1}-\epsilon_{-p_2})t'}\sin\left(\frac{2\pi p_2}{L}\right)\cot\left(\frac{\pi p_1}{L}\right)\right.\\ &\left.+\sum_{p_1,p_2}\mathrm{e}^{(2\pi\mathrm{i}/L)(p_1-p_2)(j-k)-(\epsilon_{-p_1}+\epsilon_{p_2})t-(\epsilon_{p_1}-\epsilon_{p_2})t'}\sin\left(\frac{2\pi p_1}{L}\right)\cot\left(\frac{\pi p_1}{L}\right)\right\}.\end{aligned} \tag{30}$$

In the thermodynamic limit $L\to\infty$, we have for $t' = t$

$$\begin{aligned}\langle s|\hat{n}_j(t)\hat{n}_k(t)|\tilde{\Phi}\rangle &= (1-\mathrm{e}^{-4\beta})^2\langle s|\hat{n}_j(t)\hat{n}_k(t)|1/2\rangle^{\mathrm{even}} + \mathrm{e}^{-4\beta}(1-\mathrm{e}^{-4\beta})\\ &\times\{\tfrac{1}{2}\mathrm{e}^{-2t}(I_0(2t)+I_1(2t))\delta_{j,k} + \tfrac{1}{4}[\theta(j-k)-\theta(k-j)]\\ &\times\mathrm{e}^{-2t}(I_{j-k-1}(2t)-I_{j-k+1}(2t))\}\end{aligned} \tag{31}$$

where we have used the integral representation of the modified Bessel functions $I_j(2t)$ and $\theta(x)$ is the Heaviside step function. The term $\theta(j-k)-\theta(k-j)$ is the Fourier transform of $\cot(\pi p/L)$ [18]. In particular, when $t = 0$, this reduces to

$$\langle s|\hat{n}_j\hat{n}_k|\tilde{\Phi}\rangle = \tfrac{1}{2}(1-\mathrm{e}^{-4\beta})\delta_{j,k} + \tfrac{1}{4}(1-\mathrm{e}^{-4\beta})^2(1-\delta_{j,k}) + \tfrac{1}{4}\,\mathrm{e}^{-4\beta}(1-\mathrm{e}^{-4\beta})(\delta_{j,k+1}+\delta_{j,k-1}) \tag{32}$$

where we have used the fact that there are no correlations in the state $|1/2\rangle^{\mathrm{even}}$ at $t = 0$. One clearly recognizes in the first two terms the contribution of the unconnected part of the correlation function. But the third term indeed confirms that even at $t = 0$ there are short-range correlations. This term is zero when we take $\beta\to\infty$ and we just obtain the trivial result for the $|1/2\rangle^{\mathrm{even}}$ state.

If we use equation (30) to calculate the density–density correlation function in the thermodynamic limit we see that the second term of (30) will vanish. This leaves us with a term with an amplitude proportional to $(1-m^2)^2$. The ratio of this correlation function with the square of the density (28) is independent of m in agreement with the general results known from the renormalization group approach (see for example [20]). So despite the fact that the amplitudes of the various correlation functions are non-universal as emphasized above, their ratios obey the universality hypothesis.

The results discussed above show that this approach not only allows us to recover the known results, but also provides a convenient way to compute higher-order correlation functions that can of course be translated back to the Glauber–Ising language.

5. Conclusions

We investigated the relation between the Glauber–Ising model at zero temperature and the diffusion–annihilation model in the free fermion case. We obtained the following new results.

(i) The duality transformation between the two models can be formulated as a similarity transformation if one uses a Hamiltonian with sector-dependent toroidal boundary conditions. The transformation laws for the operators are explicitly given. We also obtain the transformation laws for the states. This permits a *one-to-one* correspondence between a state of Glauber–Ising dynamics and a state of the diffusion–annihilation problem. In particular, an uncorrelated random initial state in Glauber dynamics transforms to a state with short-range correlations.

(ii) Using the free fermion solution of the diffusion–annihilation problem we have computed the time-dependent behaviour of the density and equal-time, two-point correlation function in this short-range correlated state. For the density, we recover the results of the literature. We show that, surprisingly, the presence of correlations extending over only one lattice site in the initial state leads to a long-time behaviour of the density dependent on the initial condition. The field theoretic approach we have used can be applied to the study of higher-order correlation functions. Its use depends on the form (22) of the initial state which reflects the translation invariance of this state. An initial state with the Wick's decomposition property was also discussed by Balboni *et al* [21]. They have considered a continuous system described by a boson field theory. The initial state which they have chosen is characterized by the fact that the connected correlation functions of the density operator of order higher than two vanish. They found that for pure annihilation the amplitudes are also non-universal. In the case of the initial state (14) the higher-order-connected correlation functions of the density operator are non-zero, which shows that this state has a different structure.

Acknowledgments

I am grateful to G Schütz for having introduced me to the concept of duality transformation and associated Temperley–Lieb and Hecke algebras and for carefully reading the manuscript. It is also a pleasure to acknowledge many fruitful discussions with Robin Stinchcombe, Michel Droz, Zoltan Racz, Haye Hinrichsen and John Cardy. I also thank Tim Newman for having called my attention to the universality of the ratios of amplitudes of correlation functions. The author is supported by the grant PRAXIS XXI/BD/3733/94 - JNICT - PORTUGAL.

References

[1] Glauber R J 1963 *J. Math. Phys.* **4** 294
[2] Racz Z 1985 *Phys. Rev. Lett.* **55** 1707
[3] Alcaraz F C, Droz M, Henkel M and Rittenberg V 1994 *Ann. Phys., NY* **230** 250
[4] Schütz G M 1995 *J. Phys. A: Math. Gen.* **28** 3405
[5] Family F and Amar J 1991 *J. Stat. Phys.* **65** 1235
[6] Spouge J L 1988 *Phys. Rev. Lett.* **60** 871
[7] Torney D C and McConnell H M 1983 *J. Phys. Chem.* **87** 1941
[8] Peschel I and Emery V J 1981 *Z. Phys.* B **43** 241
[9] Alcaraz F C and Rittenberg V 1993 *Phys. Lett.* **314B** 377
[10] Kadanoff L P and Swift J 1968 *Phys. Rev.* **165** 310

[11] Temperley H N V and Lieb E H 1971 *Proc. R. Soc.* A **322** 25
[12] Levy D 1991 *Phys. Rev. Lett.* **67** 1971
[13] Schütz G M 1993 *J. Phys. A: Math. Gen.* **26** 4555
[14] Henkel M, Orlandini E and Schütz G M 1995 *J. Phys. A: Math. Gen.* **28** 6335
[15] Krebs K, Pfannmüller M P, Wehefritz B and Hinrichsen H 1995 *J. Stat. Phys.* **78** 1429
[16] Simon H 1995 *J. Phys. A: Math. Gen.* **28** 6585
[17] Jordan P and Wigner E 1928 *Z. Phys.* **47** 631
[18] Santos J E, Schütz G M and Stinchcombe R B 1996 *J. Chem. Phys.* **105** 2399
[19] Bogoliubov N N 1958 *Nuovo Cimento* **7** 794
[20] Cardy J 1996 *Scaling and Renormalisation in Statistical Physics* (Cambridge: Cambridge University Press)
[21] Balboni D, Rey P and Droz M 1995 *Phys. Rev.* E **52** 6220